Baire category properties of some function spaces

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Some function spaces

For a Tychonoff space X let
$C_p(X) \subset \mathbb{R}^X$ be the space of continuous functions on X and
$B_1(X) \subset \mathbb{R}^X$ be the space of functions of the first Baire class on X.
So, each $f \in B_1(X)$ is the limit in $\mathbb{R}^X$ of some sequence $\{f_n\}_{n\in\omega} \subset C_p(X)$.

Example

- The function $\mathrm{sign} : \mathbb{R} \to \{-1, 0, 1\}$ is discontinuous but is of the first Baire class.
- The Dirichlet function, i.e., the characteristic function $\chi_{\mathbb{Q}} : \mathbb{R} \to \{0, 1\}$ of the set $\mathbb{Q}$ is not of the first Baire class.

Why?

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Characterization of functions of the first Baire class

Theorem (Lebesgue, Hausdorff, Banach)

A function $f : X \to \mathbb{R}$ on a normal space X is of the first Baire class if an only if X is F_σ-measurable in the sense that for any open set $U \subset \mathbb{R}$ the preimage $f^{-1}(U)$ is of type F_σ in X.

Q-spaces

Trivial Fact: A topological space X is discrete iff $C_p(X) = \mathbb{R}^X$.

Problem: For which spaces X we have $B_1(X) = \mathbb{R}^X$?

Definition

A topological space X is called a *Q-space* if each subset of X is of type F_σ in X.

Example: The space $\mathbb{Q}$ of rationals is a Q-space.

Fact: Under $2^{\omega_1} > \mathfrak{c}$ (which holds under CH), each second-countable Q-space is countable.

Corollary (of Lebesgue–Hausdorff–Banach)

A normal space X is a Q-space if and only if $B_1(X) = \mathbb{R}^X$.

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Main Problem

Corollary

If a normal space X is a Q-space, then $B_1(X) = \mathbb{R}^X$ and hence $B_1(X)$ is a Baire space.

Problem (Gabriyelyan)

Characterize Tychonoff spaces X having Baire $B_1(X)$.

We recall that a topological space X is *Baire* if for any sequence $(U_n)_{n\in\omega}$ of open dense sets in X the intersection $\bigcap_{n\in\omega} U_n$ is dense in X.

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The *Banach–Mazur* (or else *Choquet*) game $\mathsf{G}_{\mathsf{EN}}(X)$ on a topological space X is played by two players: E and N (abbreviated from Empty and Nonempty).

Player E starts the game choosing a non-empty open set $V_1 \subset X$ and player N responds selecting a nonempty open set $W_1 \subset V_1$.
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Otherwise the player N wins the game $\mathsf{G}_{\mathsf{EN}}(X)$.

The game $\mathsf{G}_{\mathsf{NE}}(X)$ differs from the game $\mathsf{G}_{\mathsf{EN}}(X)$ by the order of players: the player N starts the game and plays first in each inning.

Banach–Mazur game

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Oxtoby characterization of Baire and meager spaces

Theorem (Oxtoby)

A topological space X is

- *Baire iff player* E *has no wining strategy in the game* $\mathsf{G}_{\mathsf{EN}}(X)$*;*
- *meager iff player* E *has a winning strategy in* $\mathsf{G}_{\mathsf{NE}}(X)$*.*

Definition

A topological space X is *Choquet* if player N has a winning strategy in X.

Theorem (Choquet)

A metrizable space X is Choquet if and only if X is almost Čech-complete, *i.e., X contains a dense Čech-complete subspace.*

For any topological space X we have the implications:

almost Čech-complete $\Rightarrow$ Choquet $\Rightarrow$ Baire.

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Another Main Problem

It is known that for any set X the space $\mathbb{R}^X$ is Choquet.

Corollary

For any normal Q-space X the space $B_1(X) = \mathbb{R}^X$ is Choquet.

Problem

Characterize Tychonoff spaces X whose space $B_1(X)$ is Choquet.

Theorem (Banakh–Hryniv, 2018)

A topological group X is Choquet if and only if its Raikov completion $\bar{X}$ is Choquet and X is G_δ-dense in $\bar{X}$.

A subset X of a topological space $\bar{X}$ is called G_δ-dense in $\bar{X}$ if X has nonempty intersection with any non-empty G_δ-set $G \subset \bar{X}$.

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Definition (Kuratowski)

A topological space X is called a *λ-space* if each countable subset in X is of type G_δ in X.

It is clear that each Q-space is a λ-space.
But in contrast to Q-spaces, uncountable λ-spaces do exist in ZFC.

Theorem (B.-Gabriyelyan)

For a normal space X of countable pseudocharacter the following conditions are equivalent:

1. *$B_1(X)$ is a Choquet space;*
2. *$B_1(X)$ is G_δ-dense in $\mathbb{R}^X$;*
3. *X is a λ-space.*

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So, for any metrizable space X we have the implications:

$$\begin{array}{ccccc}
X \text{ is a } Q\text{-space} & \Longrightarrow & X \text{ is a } \lambda\text{-space} & \Longrightarrow & {\color{red}???} \\
\Updownarrow & & \Updownarrow & & \Updownarrow \\
B_1(X) = \mathbb{R}^X & \Longrightarrow & B_1(X) \text{ is Choquet} & \Longrightarrow & B_1(X) \text{ is Baire}
\end{array}$$

The Γ-separation game $SG_\Gamma(X)$

Let Γ be a family of subsets of a topological space X. Two sets $A, B \subset X$ are called Γ-separated if there are disjoint sets $\tilde{A}, \tilde{B} \in \Gamma$ such that $A \subset \tilde{A}$ and $B \subset \tilde{B}$.

The *Γ-separation game* $SG_\Gamma(X)$ of a topological space X is played by two players:

S and N (abbreviated from Separating and Non-separating).

Player N starts the game selecting a finite set $F_1 \subset X$ and player S responds selecting two disjoint finite sets $A_1, B_1 \subset X \setminus F_1$. At the n-th inning player N sects a finite set $F_n \subset X$ containing $F_{n-1} \cup A_{n-1} \cup B_{n-1}$ and player S selects two disjoint finite sets $A_n, B_n \subset X \setminus F_n$.

At the end of the game the player S is declared a winner if the countable sets $\bigcup_{n\in\omega} A_n$ and $\bigcup_{n\in\omega} B_n$ are Γ-separated. Otherwise player N wins the game $SG_\Gamma(X)$.

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Implications of the Baireness of function spaces

Theorem (B.-Gabriyelyan, 2019)

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1. *If $C_p(X)$ is Baire, then player* N *has no winning strategy in the* cl*-separation game* $\mathsf{SG}_{\mathsf{cl}}(X)$.
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Implications of the Baireness of function spaces

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Let $\mathcal{B}$ be a family of subsets of a topological space X.
A subset $D \subset X$ is called $\mathcal{B}$-dense if $D \cap B \neq \emptyset$ for any $B \in \mathcal{B}$.
Observe that a subset D of a topological space X is dense in X if and only if it is $\mathcal{B}$-dense for some/any π-base $\mathcal{B}$ for X.
A family $\mathcal{B}$ of nonempty open sets of a topological space X is called a π-base for X if each nonempty open set $U \subset X$ contains some set $B \in \mathcal{B}$.

Fact:

For any π-base $\mathcal{B}$ in a Baire space X, any $\mathcal{B}$-dense G_δ-sets $A, B \subset X$ have nonempty intersection.

Definition

A topological space X is called *airy* if there exists a countable family $\mathcal{B}$ of infinite sets in X such that any $\mathcal{B}$-dense G_δ-sets $A, B \subset X$ have nonempty intersection $A \cap B \neq \emptyset$.

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The G_δ-separation game in an airy space

Theorem (B.-Gabriyelyan)

If a topological space X is airy, then player N has a winning strategy in the G_δ-separation game on X.

Corollary

If for a topological space X the function space $B_1(X)$ is Baire, then the space X is non-airy.

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Airy versus universally meager spaces

Definition (between Todorcevic and Zakrzewski)

A topological space X is called *universally meager* if for any Baire space B having a countable π-base and any continuous map $f : B \to X$ there exists a nonempty open set $U \subset B$ whose image $f(U)$ is finite.

Theorem

Each non-airy space X is universally meager.

Proof.

To be presented by vaiwing hands. □

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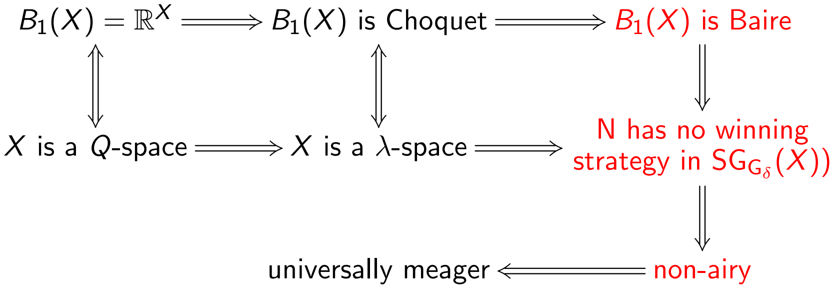
The final diagram



Problem

Find examples distinguishing red properties from the last column.

Example (Zdomskyy, 2019)

Under $\mathfrak{b} = \mathfrak{c}$ there exists an airy universally meager subset $X \subset \mathbb{R}$.

The final diagram

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\begin{array}{ccccc}
B_1(X) = \mathbb{R}^X & \Longrightarrow & B_1(X) \text{ is Choquet} & \Longrightarrow & \color{red}{B_1(X) \text{ is Baire}} \\
\Updownarrow & & \Updownarrow & & \Downarrow \\
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T.Banakh, S.Gabriyelyan,
Baire category properties of some Baire type function spaces,
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Thanks for your attention!

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Happy Birthdays!