Compactly supported analytic P-ideals

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joint work with **Barnabas Farkas**

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- $\mathcal{I}$ can be treated as a subset of 2^ω(via $A \mapsto \chi_A$);
- $\mathcal{I}$ is a P-ideal if for each (A_n) from $\mathcal{I}$, there is $A \in \mathcal{I}$ such that $A_n \subseteq^* A$ for every n.

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- Both $\mathrm{Fin}(\varphi)$ and $\mathrm{Exh}(\varphi)$ are analytic P-ideals.
- **Theorem (Solecki)** For every analytic P-ideal there is an LSC submeasure φ such that

$$\mathcal{I} = \mathrm{Exh}(\varphi).$$

How to generate ideals from families of finite sets

Let $\mathcal{F}$ be a family of finite subsets of ω (covering ω). Assume that $\mathcal{F}$ is hereditary, i.e. $F \in \mathcal{F}$ whenever $F \subseteq G$ for some $G \in \mathcal{F}$.

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Define

$$\varphi_{\mathcal{F},x}(A) = \sup\{\sum_{n \in F} x_n \colon F \in \mathcal{F}\}$$

and

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$$\lambda = \big(1, 1/2, 1/2, \underbrace{1/4, \cdots, 1/4}_{\text{4 times}}, \underbrace{1/8, \cdots, 1/8}_{\text{8 times}}, \cdots\big).$$

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$$\varphi_{\mathcal{F}}(A) = \sup\{\sum_{n \in F} \lambda_n \colon F \in \mathcal{F}\}$$

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$$\mathcal{I}_{\mathcal{F}} = \mathrm{Exh}(\varphi_{\mathcal{F}}).$$

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$$\varphi_{\mathcal{F}}(A) = \sup\{\sum_{n \in F} \lambda_n \colon F \in \mathcal{F}\}$$

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- $\mathcal{F} = \{F \in [\omega]^{<\omega} \colon |F \cap [2^n, 2^{n+1})| < 2^n/n\}^{\downarrow}$ $\quad \mathcal{I}_{\mathcal{F}}$ -Farah's ideal.

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Assume $\mathcal{F}$ is compact (as a subset of 2^ω) and $x \in [0, \infty)^\omega$. If $\mathcal{I}_{\mathcal{F},x}$ is non-trivial, then it is not F_σ.

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- and then the proof starts.

□

Application: DU problem

Theorem

Let μ be a measure on ω such that $\mu(\{n\}) \to 0$ and $\mu(\omega) = \infty$. Assume $\mathcal{F}$ is hereditary, covers ω and is such that for each $A \in [\omega]^{<\omega}$ there is $F \in \mathcal{F}$ such that $F \subseteq A$ and

$$\mu(F) > \mu(A)/2.$$

Then there is $N \in [\omega]^{\omega}$ such that $[N]^{<\omega} \subseteq \mathcal{F}$.

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- Assume $\mathcal{F}$ is as above, but there is no *homogenuous* N.
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- $\mathcal{I}_{\mathcal{F},\mu} = \mathrm{Fin}(\mu)$.
- $\mathcal{I}_{\mathcal{F},\mu}$ is F_σ. Contradiction.

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Application: Mazur's Lemma

Theorem (Mazur's Lemma)

Let X be a Banach space and let (x_n) be a bounded weakly null sequence in X. Then for each $\varepsilon > 0$ there is a finite convex combination $y = \sum_i \alpha_i x_i$ such that $\|y\| < \varepsilon$.

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Theorem (Mazur's Lemma +)

Let X be a Banach space, (x_n) be a bounded weakly null sequence in X, and let μ be a measure on ω such that $\mu(\omega) = \infty$ and $\mu(\{n\}) \to 0$. Then for each $\varepsilon > 0$ there is a finite $G \subseteq \omega$ and a convex combination $y = \sum_{i \in G} \alpha_i x_i$ where $\alpha_i = \mu(\{i\})/\mu(G)$, such that $\|y\| < \varepsilon$.

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One can define recursively Schreier families of higher order: $\mathcal{S}_\alpha$, $\alpha < \omega_1$, e.g.

$$\mathcal{S}_2 = \{\bigcup_{j \leq n} F_j \colon F_0 < \cdots < F_n \in \mathcal{S}, n \leq \min F_0 + 1\}$$

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Theorem

For each $\alpha < \omega_1$

$$\mathcal{I}_{\mathcal{S}_{\alpha+1}} = \mathrm{Exh}(\varphi_{\mathcal{S}_{\alpha+1}}) \subseteq \mathrm{Fin}(\varphi_{\mathcal{S}_{\alpha+1}}) \subseteq \mathrm{Exh}(\varphi_{\mathcal{S}_\alpha}) = \mathcal{I}_{\mathcal{S}_\alpha}.$$

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We call $\mathcal{I}_{\mathcal{S}_\alpha}$'s *Schreier ideals*. Are they pairwise different?

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for each α we have $\mathcal{I}_{\mathcal{S}_{\alpha+1}} \subsetneq \mathcal{I}_{\mathcal{S}_\alpha}$.

Thanks.