About porosity in the space of cliquish functions

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1931, **S. Banach** - the set of nowhere diferentiable functions is residual in the space of continuous functions under the supremum metric.

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Darboux-like functions in relation to dynamical systems:

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2. K. L. Kellum, *Iterates of almost continuous functions and Sharkovskii's theorem*, Real Anal. Exchange **14** (1988/1989), 420–422.
3. H. Pawlak, R. Pawlak, *Dynamics of Darboux functions*, Tatra Mt. Math. Publ. **42** (2009), 51–60.
4. H. Pawlak, R. Pawlak, *Transitivity, dense orbits and some topologies finer than the natural topology of the unit interval*, Tatra Mt. Math. Publ. **35** (2007), 1–12.
5. R. Pawlak, *On Sharkovsky's property of Darboux functions*, Tatra Mt. Math. Publ. **42** (2009), 95–105.
6. R. Pawlak, *On the entropy of Darboux functions*, Colloq. Math. **116** (2) (2009), 227–241.

Porosity in spaces of Darboux-like functions:

1. J. Kucner, R. Pawlak, B. Świątek, *On small subsets of the space of Darboux functions*, Real Anal. Exchange **25** (1) (1999), 343–358.
2. H. Rosen, *Porosity in spaces of Darboux-like functions*, Real Anal. Exchange **26** (1) (2000), 195–200.

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2. H. Rosen, *Porosity in spaces of Darboux-like functions*, Real Anal. Exchange **26** (1) (2000), 195–200.

Porosity in spaces of functions:

1. M. Filipczak, G.I., J. Wódka, *Comparison of some families of real functions in porosity terms*, Math. Slovaca, **67** (5) (2017), 1155–1164.
2. G.I., A. Karasińska, E. Wagner-Bojakowska, *Some remarks on quasi-continuous functions*, Tatra Mt. Math. Publ. **65** (2016), 151–159.
3. G.I., E. Wagner-Bojakowska, *On some modification of Darboux property,* Math. Slovaca **66** (1) (2016), 79–88.
4. G.I., E. Wagner-Bojakowska, *On some subclasses of the family of Darboux Baire 1 functions*, Opuscula Math. **34** (4) (2014), 777–788.
5. G.I., E. Wagner-Bojakowska, *On some subfamilies of Darboux quasi-continuous functions*, Bull. Soc. Sci. Lett. Łódź Sér. Rech. Déform. **64** (3) (2014), 31–43.

$f : \mathbb{R} \to \mathbb{R}$

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S. Kempisty, *Sur les fonctions quasicontinues*, Fund. Math. **19** (1932), 184–197.

Definition (S. Kempisty, 1932)

A function f is **quasi-continuous at a point** x if for every neighbourhood U of x and for every neighbourhood V of $f(x)$ there exists a non-empty open set $G \subset U$ such that $f(G) \subset V$.

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$\mathcal{Q}$ - the family of all quasi-continuous functions

H. P. Thielman, *Types of functions*, Amer. Math. Monthly **60** (1953), 156–161.

Definition (H. P. Thielman, 1953)

A function f is **cliquish at a point** x if for every neighbourhood U of x and for each $\epsilon > 0$ there exists a non-empty open set $G \subset U$ such that $| f(y) - f(z) | < \epsilon$ for each $y, z \in G$.

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$\mathcal{C}_q$ - the family of all cliquish functions

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$\mathcal{B}$a - the family of functions having the Baire property

$\mathcal{Q} \subset \mathcal{C}_q$

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Theorem (Oxtoby)

A function f has the Baire property iff there exists a set P of the first category such that $f\,|_{\mathbb{R}\setminus P}$ is continuous.

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T. Mańk, T. Świątkowski, *On some class of functions with Darboux's characteristic*, Zeszyty Nauk. Politech. Łódz. Mat. **11** (301) (1977), 5–10.

Definition (T. Mańk, T. Świątkowski, 1977)

A function f has the **Świątkowski property** if for each interval $(a, b) \subset \mathbb{R}$ there exists a point $x \in (a, b)$ such that f is continuous at x and $f(x)$ is situated between $f(a)$ and $f(b)$.

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$\mathcal{S}$ - the family of functions having the Świątkowski property

$$\mathcal{S} \subset \mathcal{C}_q \subset \mathcal{B}a$$

Let $f, g \in \mathcal{B}a$.
Put

$$\rho(f,g) = min\left\{1, sup\left\{\mid f(t) - g(t) \mid : t \in \mathbb{R}\right\}\right\}.$$

Let (X, ρ) - metric space.

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$$p(M, x) = 2 \limsup_{r \to 0^+} \frac{\gamma(x, r, M)}{r}.$$

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L. Zajiček, *On σ-porous sets in abstract spaces*, Abstr. Appl. Anal. **5** (2005), 509–534.

Definition (L. Zajiček)

A set $M \subset X$ is **porous (strongly porous)** if $p(M, x) > 0$ ($p(M, x) = 1$) for each $x \in M$.

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G.I., E. Wagner-Bojakowska, *Porous subsets in the space of functions having the Baire property*, Math. Slovaca **67** (6) (2017), 1333-1344.

Fix $q > 0$.

Definition

A set $M \subset X$ is **at least q-porous** if $p(M) \geq q$.

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A set $M \subset X$ is **at least** q-**porous** if $p(M) \geq q$.

Definition

A set $M \subset X$ is q-**porous** if $p(M) = q$.

M. Filipczak, G.I., J. Wódka

M. Filipczak, G. Ivanova, J. Wódka, *Comparison of some families of real functions in porosity terms,* Math. Slovaca **67** (5) (2017), 1155-1164.

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Theorem

The family $\mathcal{QS}$ is at least $1/2$-porous in $(\mathcal{Q}, \rho)$.

M. Filipczak, G.I., J. Wódka

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Theorem

The family $\mathcal{QS}$ is not $3/4$-porous in $(\mathcal{Q}, \rho)$.

M. Filipczak, G.I., J. Wódka

$\mathcal{QS}$ <u>is</u> porous in $(\mathcal{Q}, \rho)$

$\mathcal{QS}$ <u>is not</u> strongly porous in $(\mathcal{Q}, \rho)$

M. Filipczak, G.I., J. Wódka

$\mathcal{QS}$ is porous in $(\mathcal{Q}, \rho)$

$\mathcal{QS}$ is not strongly porous in $(\mathcal{Q}, \rho)$

Open question:

Does there exist $q \in [1/2, 3/4)$ such that $\mathcal{QS}$ is exactly q-porous in $(\mathcal{Q}, \rho)$?

G.I., E. Wagner-Bojakowska

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$$\mathcal{S} \subset^{?} \mathcal{C}_q \subset^{sp} \mathcal{D}\mathcal{B}a$$

Theorem

The set $\mathcal{S}$ is at least $2/3$-porous in $(\mathcal{C}_q, \rho)$.

Lemma

Let $f \in \mathcal{C}q$ *and* $\varepsilon > 0$. *Then there exist sequences of disjoint intervals* $\{(a_n, b_n)\}_{n\in\mathbb{N}}$ *and* $\{\left(y_k^n, y_{k+1}^n\right)\}_{k\in\mathbb{Z}}$, $n \in \mathbb{N}$, *such that:*

(i) *for each* $n \in \mathbb{N}$

$$(a_n, b_n) = \bigcup_{k\in\mathbb{Z}} \left[y_k^n, y_{k+1}^n\right];$$

(ii) $osc\left(f, \left[y_k^n, y_{k+1}^n\right]\right) < \epsilon$ *for each* $n \in \mathbb{N}$, $k \in \mathbb{Z}$;

(iii) *for each* $n \in \mathbb{N}$ *we have* $\lim\limits_{n\to\infty} y_k^n = b_n$ *and* $\lim\limits_{n\to-\infty} y_k^n = a_n$;

(iv) *the set* $\mathbb{R} \setminus \bigcup_{n\in\mathbb{N}} (a_n, b_n)$ *is contained in* $D(f)$ *and nowhere dense.*

Suppose that I and J are intervals.

Definition

A function f is a (I, J)-*left side surjective* if for all $t \in I$ we have $f((\inf I, t)) = J$. Analogously, we say that f is a (I, J)-*right side surjective function* if for all $t \in I$ we have $f((t, \sup I)) = J$. A function f is a (I, J)-*bi-surjective function* if it is both left and right side surjective.

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Theorem

For each $q \in (2/3, 1)$ the set $\mathcal{S}$ not q-porous in $(\mathcal{C}_q, \rho)$.

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The set $\mathcal{S}$ is 2/3-porous in $(\mathcal{C}_q, \rho)$.

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Theorem

The set $\mathcal{S}$ is 2/3-porous in $(\mathcal{C}_q, \rho)$.

$$\mathcal{S} \subset^{2/3-p} \mathcal{C}_q \subset^{sp} \mathcal{DB}a$$

M. Filipczak, G.I., J. Wódka

Open question:

Does there exist $q \in [1/2, 3/4)$ such that $\mathcal{QS}$ is exactly q-porous in $(\mathcal{Q}, \rho)$?

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G.I., A. Karasińska

G.I., A. Karasińska, *About porosity of some Świątkowski functions in the space of quasi-continuous functions*, submited

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