On Cantor Spaces

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Outline

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PRELIMINARY DEFINITIONS

DEFINITION

A topological space is a *Cantor space* if it is non-empty, perfect, compact, totally disconnected, and metrizable.

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BROUWER'S THEOREM

Each Cantor space is homeomorphic to the Cantor ternary set.

Preliminary definitions

Definition

A set $E \subset \mathbb{R}^n$ is *microscopic* if for each $\epsilon > 0$ there exists a sequence of rectangles $\{I_k\}_{k \in \mathbb{N}}$ such that

$$E \subseteq \cup_{k \in \mathbb{N}} I_k \textit{ and } \lambda(I_k) \leq \epsilon^k, \textit{ for } k \in \mathbb{N},$$

where λ is the Lebesgue measure on $\mathbb{R}^n$.

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Definition

A set $E \subset \mathbb{R}^n$, $n \geq 2$, is *strongly microscopic* if for each $\epsilon > 0$ there exists a sequence of cubes $\{I_k\}_{k \in \mathbb{N}}$ such that

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Example [A. Karasińska, E. Wagner-Bojakowska, 2014]
Let $A = [0,1] \times \{0\} \subset \mathbb{R}^2$, $\epsilon > 0$. Let $I_1 = [0,1] \times [-\frac{\epsilon}{3}, \frac{\epsilon}{3}]$ and $I_k = \emptyset$ for $k > 1$. Hence, $A \subset \cup_{k\in\mathbb{N}} I_k$ and $\lambda(I_k) < \epsilon^k$ $\forall k \in \mathbb{N}$. Then, A is microscopic. Suppose that A is strongly microscopic. Let $\epsilon = \frac{1}{16}$. Then, there exists $\{I_k\}_{k\in\mathbb{N}}$ squares with sides of length a_k s.t. $A \subset \cup_{k\in\mathbb{N}} I_k$ and $\lambda(I_k) < \left(\frac{1}{16}\right)^k$ $\forall k \in \mathbb{N}$. Hence, $a_k < \left(\frac{1}{4}\right)^k$ and $1 \leq \sum_{k=1}^{\infty} I_k < \sum_{k=1}^{\infty} \left(\frac{1}{4}\right)^k = \frac{1}{3}$, which is a contraddiction. So A is not strongly microscopic.

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4. Every countable union of (strongly, resp.) microscopic sets is (strongly, resp.) microscopic.
5. Every strongly microscopic set E has α-dimensional Hausdorff measure equal to zero for all $\alpha > 0$, and thus it has Hausdorff dimension zero.

Outline

The space $(\mathcal{K}, \mathcal{H})$

Let $n \geq 1$. Consider the space $(\mathcal{K}, \mathcal{H})$, where $\mathcal{K}$ consists of the nonempty, compact subsets of $[0, 1]^n$ furnished with the Hausdorff metric given by

$$\mathcal{H}(A, E) = \inf\{\delta > 0 : A \subset B_\delta(E), E \subset B_\delta(A)\}.$$

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These properties hold in $[0,1]$ ([A.M. Bruckner, J.B. Bruckner, B.S. Thomson, 1997]) and also in $[0,1]^n$ ([E. D'Aniello, T.H. Steele, 2015]).

The space $(\mathcal{K}, \mathcal{H})$

Definition

Let $I_1, \dots, I_t$ be open intervals (relative to $[0, 1]^n$). Let $B(I_1, \dots, I_t)$ be the collection of all $K \in \mathcal{K}$ s.t.:

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2. $K \cap I_i \neq \emptyset$ for each $i \in \{1, \dots, t\}$.

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Lemma [E. D'Aniello, M., 2019]

Let $I_1, ..., I_t$ be open intervals (relative to $[0, 1]^n$). Then $B(I_1, ..., I_t)$ is open in $(\mathcal{K}, \mathcal{H})$.

Strongly microscopic compact sets

Theorem [E. D'Aniello, M., 2019]

The typical compact subset of $[0, 1]^n$ is a strongly microscopic set.

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Sketch of the proof

Let

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that is $\mathcal{M}_{\mathcal{SK}} = \mathcal{K} \cap \mathcal{M}_{\mathcal{S}}$, where $\mathcal{M}_{\mathcal{S}}$ is the family of all strongly microscopic subsets of $\mathbb{R}^n$.

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that is $\mathcal{M}_{\mathcal{S}\mathcal{K}} = \mathcal{K} \cap \mathcal{M}_{\mathcal{S}}$, where $\mathcal{M}_{\mathcal{S}}$ is the family of all strongly microscopic subsets of $\mathbb{R}^n$. Then:

- $\mathcal{M}_{\mathcal{S}\mathcal{K}}$ is dense in $\mathcal{K}$.
- $\mathcal{M}_{\mathcal{S}\mathcal{K}}$ is a G_δ subset of $\mathcal{K}$.
  Indeed $\mathcal{M}_{\mathcal{S}\mathcal{K}} = \cap_{s=1}^{\infty} \mathcal{K}^{[s]}$ where, for each $s \in \mathbb{N}$, $\mathcal{K}^{[s]}$ is the collection of all $E \in \mathcal{K}$ s.t.

  $$\exists \{I_j\}_{j \in \mathbb{N}} \text{ sequence of open cubes with } E \subseteq \cup_{j \in \mathbb{N}} I_j, \lambda(I_j) \leq \left(\frac{1}{s}\right)^j,$$

  and each $\mathcal{K}^{[s]}$ is open.

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- On the other hand, on the real line, there exist symmetric Cantor spaces that are microscopic.
- More is true: microscopic symmetric Cantor spaces are a residual family ([M. Balcerzak, T. Filipczak, P. Nowakowski, 2019]).

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Question 1:
How frequent are strongly microscopic Cantor spaces or, more generally, microscopic Cantor spaces?

Strongly microscopic Cantor spaces

Let $\mathcal{IR} = \{(a_1, ..., a_n) \in [0, 1]^n : a_j \in [0, 1] \setminus \mathbb{Q},$ for all $1 \leq j \leq n\}$ that is, $\mathcal{IR}$ denotes the collection of all points in $[0, 1]^n$ with all the coordinates irrational.

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Consider

$$\mathcal{K}_1 = \{F \in \mathcal{K} : F \text{ is a Cantor space and } F \subseteq \mathcal{I}R\} .$$

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Consider

$$\mathcal{K}_1 = \{F \in \mathcal{K} : F \text{ is a Cantor space and } F \subseteq \mathcal{IR}\}.$$

Theorem [E. D'Aniello, T.H. Steele, 2015]

The collection $\mathcal{K}_1$ is a dense set of type G_δ in $\mathcal{K}$.

STRONGLY MICROSCOPIC CANTOR SPACES

THEOREM [E. D'ANIELLO, M., 2019]

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The collection of the non-empty, compact and strongly microscopic subsets of $[0,1]^n$, that is $\mathcal{M}_{\mathcal{SK}}$, is a dense set of type G_δ.

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Proof

The collection of the non-empty, compact and strongly microscopic subsets of $[0,1]^n$, that is $\mathcal{M}_{\mathcal{SK}}$, is a dense set of type G_δ.
The collection $\mathcal{K}_1$ also is a dense G_δ set. Since the intersection of two dense G_δ sets is still a dense G_δ set, the thesis follows.

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Every strongly microscopic set $E \subset \mathbb{R}^n$ has α-dimensional Hausdorff measure equal to zero for all $\alpha > 0$, and thus it has Hausdorff dimension zero.

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Question 2: The previous implication can be reverted?
([E. D'Aniello, M., 2019])

An Example

Fix $c \geq 2^n + 1$, let $V_0 = 1$, and $V_k = \frac{1}{c^{k^2}}$ for $k \in \mathbb{N}$.

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At the first step we select 2^n disjoint cubes in $[0, 1]^n$ of measure $V_1 = \frac{1}{c}$, and having one vertex in common with $[0, 1]^n$. We list these cubes as Q_{i_1}, with $i_1 \in \{1, ..., 2^n\}$.

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At the second step, in each Q_{i_1}, we select 2^n disjoint cubes of measure $V_2 = \frac{1}{c^4}$, having one vertex in common with Q_{i_1}. We list them as Q_{i_1, i_2}, with $(i_1, i_2) \in \{1, ..., 2^n\}^2$.

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At the k-th step, in each $Q_{i_1,...,i_{k-1}}$ with $(i_1, ..., i_{k-1}) \in \{1, ..., 2^n\}^{k-1}$, we select 2^n disjoint cubes of measure $V_k = \frac{1}{c^{k^2}}$ and having one vertex in common with $Q_{i_1,...,i_{k-1}}$. We list these cubes as $Q_{i_1,...,i_k}$, $(i_1, ..., i_k) \in \{1, ..., 2^n\}^k$.

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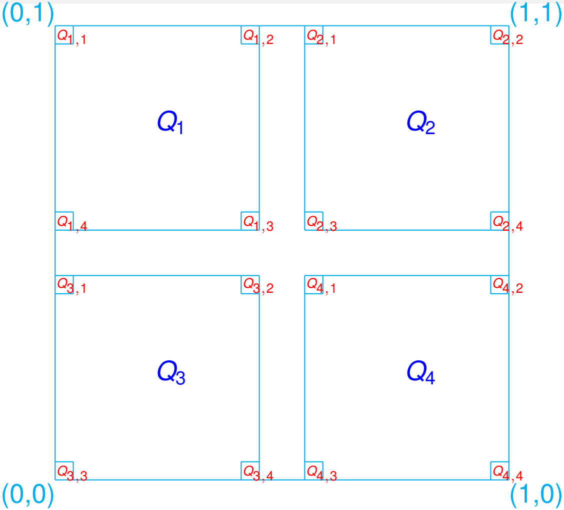

Figure: Construction of the cubes of the first two steps (for $n = 2$)

An Example

for each $k \in \mathbb{N}$, at the k-th step we have constructed a collection of cubes $\mathcal{Q}_k = \{Q_{i_1,\dots,i_k}$, with $(i_1, \dots, i_k) \in \{1, \dots, 2^n\}^k\}$ s.t.:

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1. each Q_{i_1} contains a vertex of $[0,1]^n$; for $k > 1$ each $Q_{i_1,\dots,i_k}$ has one vertex in common with $Q_{i_1,\dots,i_{k-1}}$ and $\lambda(Q_{i_1,\dots,i_k}) = V_k = \frac{1}{c^{k^2}}$;

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2. for each $(i_1, \dots, i_{k-1}, j_k), (i_1, \dots, i_{k-1}, j'_k)$, with $j_k, j'_k \in \{1, \dots, 2^n\}$, we have $dist(Q_{i_1,\dots,i_{k-1},j_k}, Q_{i_1,\dots,i_{k-1},j'_k}) \geq \frac{1}{\sqrt[n]{c^{(k-1)^2}}} - \frac{2}{\sqrt[n]{c^{k^2}}}$ and, if we set $d_k = \frac{1}{\sqrt[n]{c^{(k-1)^2}}} - \frac{2}{\sqrt[n]{c^{k^2}}}$, then d_k^n is the Lebesgue measure of the largest cube contained in $Q_{i_1,\dots,i_{k-1}} \setminus \cup_{i_k \in \{1,\dots,2^n\}} Q_{i_1,\dots,i_k}$, having inner points in common with at most one cube $Q_{i_1,\dots,i_k}$.

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3. for each $(i_1, \dots, i_k), (j_1, \dots, j_k) \in \{1, \dots, 2^n\}^k$, we have $dist(Q_{i_1,\dots,i_k}, Q_{j_1,\dots,j_k}) \geq D_k$, where

$$D_1 = d_1, \text{ for } k \geq 2, D_k = min\{d_k, D_{k-1}\},$$

and D_k^n is the Lebesgue measure of the largest cube contained in $[0,1]^n \setminus \cup_{(i_1,\dots,i_k)\in\{1,\dots,2^n\}^k} Q_{i_1,\dots,i_k}$, having inner points in common with at most one cube $Q_{i_1,\dots,i_k}$, with $(i_1, \dots, i_k) \in \{1, \dots, 2^n\}^k$.

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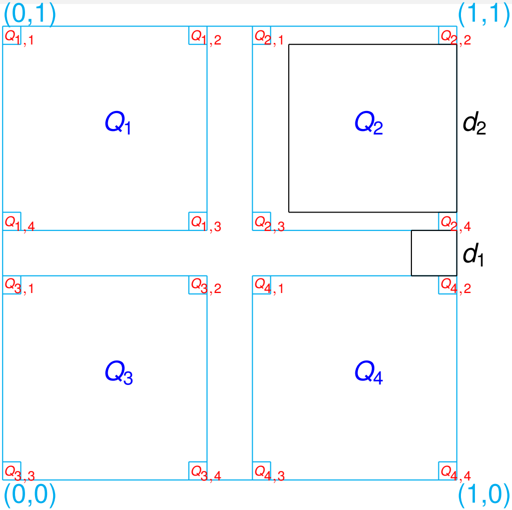

Figure: The lengths d_1 and d_2 of the first two steps (for $n = 2$)

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- $\mathcal{H}^\alpha(C) = 0$ for each $\alpha > 0$ and thus C has Hausdorff dimension 0.
- C is not microscopic and hence it is not strongly microscopic.

Thank you for your attention!

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