Mycielski among trees - category case

Marcin Michalski, Robert Rałowski, Szymon Żeberski

Wrocław University of Science and Technology

5th Real Analysis Workshop
15-16.06.2019, Konopnica

Theorem (Mycielski)

For every comeager (conull) set $X \subseteq [0,1]^2$ *there is a perfect set* $P \subseteq [0,1]$ *satisfying* $P \times P \subseteq X$.

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What about other types of trees? Goal: For a comeager set

$G \subseteq \omega^\omega \times \omega^\omega$ find trees $T_1 \subseteq T_2$ of some type satisfying $[T_1] \times [T_2] \subseteq G$, where

$$[T] = \{x \in \omega^\omega : (\forall n \in \omega)(x \upharpoonright n \in T)\}$$

Denote $\omega^{<\omega} = \bigcup_{n\in\omega} \omega^n$.

Definition

Let $T \subseteq \omega^{<\omega}$ be tree. Then

- $\mathrm{succ}_T(\sigma) = \{n \in \omega : \sigma^\frown a \in T\}$ *for each* $\sigma \in T$;
- $\mathrm{split}(T) = \{\sigma \in T : |\mathrm{succ}_T(\sigma)| \geqslant 2\}$;
- $\omega\text{-split}(T) = \{\sigma \in T : |\mathrm{succ}_T(\sigma)| = \omega\}$.
- $\mathrm{stem}(T) \in T$ *is the shortest splitting node of* T.

Definition

We call a tree $T \subseteq \omega^{<\omega}$ *a*

- *Sacks (or perfect) tree if for each* $\sigma \in T$ *there is* $\tau \in T$ *such that* $\sigma \subseteq \tau$ *and* $\tau \in \mathrm{split}(T)$*;*
- *Miller (or superperfect) tree if for each* $\sigma \in T$ *there is* $\tau \in T$ *such that* $\sigma \subseteq \tau$ *and* $\tau \in \omega\text{-}\mathrm{split}(T)$*;*
- *Laver tree if for each* $\sigma \in T$ *satisfying* $\mathrm{stem}(T) \subseteq \sigma$ *we have* $\sigma \in \omega\text{-}\mathrm{split}(T)$*.*

Definition

We call a perfect tree $T \subseteq \omega^{<\omega}$ *a*

- *uniformly perfect tree if*

$$(\forall n \in \omega)(T \cap \omega^n \subseteq \mathrm{split}(T) \ \vee \ T \cap \omega^n \cap \mathrm{split}(T) = \emptyset).$$

- *Silver tree if*

$$(\forall \sigma, \tau \in T)(|\sigma| = |\tau| \Rightarrow (\forall n \in \omega)(\sigma^\frown n \in T \Leftrightarrow \tau^\frown n \in T).$$

Laver trees are too big

Proposition

There exists a G_δ set $G \subseteq \omega^\omega$ such that $[T] \not\subseteq G$ for every Laver tree $T \subseteq \omega^{<\omega}$.

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Proof.

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- Set $G = \{x \in \omega^\omega : (\exists^\infty n \in \omega)(x(n) = 0)\}$. It is dense G_δ.
- If T is a Laver tree then

$$T' = \{x \in T : x(n) \neq 0 \text{ for } n \geqslant |\text{stem}(T)|\}$$

is a Laver subtree of T and $T' \cap G = \emptyset$. □

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Corollary

There exists a $G_\delta \subseteq \omega^\omega \times \omega^\omega$ *such that* $[T] \times \{y\} \not\subseteq G$ *for every laver tree* $T \subseteq \omega^{<\omega}$ *and* $y \in \omega^\omega$.

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Theorem

For every comeager set $G \subseteq \omega^\omega \times \omega^\omega$ there exist a uniformly perfect tree $T_1 \subseteq \omega^{<\omega}$ and a Miller tree $T_2 \subseteq \omega^{<\omega}$ such that $[T_1] \times [T_2] \subseteq G \cup \Delta$.

Miller trees are half-good

Lemma 1

For every open dense set $U \subseteq \omega^\omega \times \omega^\omega$ and two open sets $V_1, V_2 \subseteq \omega^\omega$ there are sequences $\sigma_1, \sigma_2 \in \omega^{<\omega}$ satisfying $[\sigma_1] \subseteq V_1$, $[\sigma_2] \subseteq V_2$, $|\sigma_1| = |\sigma_2|$ such that $[\sigma_1] \times [\sigma_2] \subseteq U$ and $[\sigma_2] \times [\sigma_1] \subseteq U$.

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Lemma 2

For every open dense set $U \subseteq \omega^\omega \times \omega^\omega$, a finite sequence of open sets $(V_k : 0 \leqslant k < n)$ in ω^ω there is a sequence of sequences $(\sigma_k : 0 \leqslant k < n)$ such that:

1. $[\sigma_k] \subseteq V_k$ for all $0 \leqslant k < n$,
2. $|\sigma_k| = |\sigma_l|$ for all $0 \leqslant k, l < n$,
3. $[\sigma_l] \times [\sigma_k] \subseteq U$ for all distinct $0 \leqslant k, l < n$.

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Idea of proof.

Construct a nice Miller tree using Lemma 2.

Theorem

There is an open dense set U *such that* $[T_1] \times [T_2] \not\subseteq U \cup \Delta$ *for any Miller trees* $T_1, T_2 \subseteq \omega^{<\omega}$.

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- Let $Q = \{(q_1, q_2) \subseteq \mathbb{Q}^2 : \mathrm{supp}(q_1) = \mathrm{supp}(q_2)\}$, where $\mathrm{supp}(q) = \max\{n \in \omega : q(n) \neq 0\} + 1$.

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- Set $U = \bigcup_{q \in Q}[q_1 \restriction (\mathrm{supp}(q_1) + K(q))] \times [q_2 \restriction (\mathrm{supp}(q_2) + K(q))]$, where $K(q) = \max\{q_1(n), q_2(n) : n \in \omega\}, q = (q_1, q_2)$

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- Choose $(x, y) \in [T_1] \times [T_2]$ such that $(x, y) \notin U \cup \Delta$

Proposition

There exists a comeager set $G \subseteq \omega^\omega$ such that $[T] \not\subseteq G$ for any uniformly perfect Miller tree $T \subseteq \omega^{<\omega}$.

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Thank you for your attention!

M. Michalski, R. Ratowski, Sz. Żeberski, Mycielski among trees, arXiv:1905.09069.