Semiregular matrices and associated ideals

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Joint work with Pratulananda Das and Rafał Filipów.

Semiregular matrix

Definition (Toeplitz)

A matrix $A = (a_{i,k})$ with nonnegative elements is regular if

- $\lim_{i\to\infty} a_{i,k} = 0$ for every $k \in \mathbb{N}$;
- $\lim_{i\to\infty} \sum_{k\in\mathbb{N}} a_{i,k} = 1$.

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Definition

A semiregular matrix $A = (a_{i,k})$ is

- *of type 1* if $\sum_{k\in\mathbb{N}} a_{i,k} < \infty$ for all but finitely many i;
- *of type 2* if $\sum_{k\in\mathbb{N}} a_{i,k} = \infty$ for infinitely many i;

Matrix ideals

Definition

Let $A = (a_{i,k})$ be a nonnegative matrix of either type. Then

$$\mathcal{I}(A) = \{B \subseteq \mathbb{N} : \lim_{i \to \infty} \sum_{k \in B} a_{i,k} = 0\}$$

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If $\mathcal{I} = \mathcal{I}(A)$ for some regular matrix A, then we will denote it by $\mathcal{I} \in \mathrm{REG}$.
If $\mathcal{I} = \mathcal{I}(A)$ for some semiregular matrix A of type 1 (semiregular of type 2), then we will denote it by $\mathcal{I} \in \mathrm{SR1}(\mathcal{I} \in \mathrm{SR2})$.

Examples

$\mathrm{Fin} = \{B \subseteq \mathbb{N} : B \text{ is finite}\} \in \mathrm{REG}$ as $\mathrm{Fin} = \mathcal{I}(I)$, where I is the identity matrix

$$I = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

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$\mathcal{I}_d = \{B \subseteq \mathbb{N}\colon\ \limsup_{n\to\infty} \frac{|B\cap n|}{n} = 0\} \in \mathsf{REG}$ as $\mathcal{I}_d = \mathcal{I}(C)$, where C is the Cesáro matrix

$$C = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

Examples

$\mathrm{Fin} \in \mathsf{SR1}$ as $\mathrm{Fin} = \mathcal{I}(A)$ for

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 2 & 0 & \cdots \\ 0 & 0 & 3 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

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$\mathrm{Fin} \in$ SR2 as $\mathrm{Fin} = \mathcal{I}(A)$ for

$$A = \begin{pmatrix} 1 & 1 & 1 & \cdots \\ 0 & 1 & 1 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

Properties

Theorem (Freedman-Sember)

Let $\mathcal{I} \in REG$. Then $\mathcal{I}$ is a P-ideal, i.e. for every sequence $(A_n)_{n\in\mathbb{N}}$ of sets belonging to $\mathcal{I}$ there exists a set $A \in \mathcal{I}$ such that $A_n \setminus A$ is finite for every n.

Theorem (Bartoszewicz-Das-Głąb)

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Theorem

Let $\mathcal{I} \in SR1$. Then $\mathcal{I}$ is a P-ideal.

Something new

Let $\{P_n : n \in \mathbb{N}\}$ be a partition of $\mathbb{N}$ into infinite sets. Then $\mathcal{I} = \{B \subseteq \mathbb{N} : B \cap P_n = \emptyset$ for all but finitely many $n\} \in$ SR2 and $\mathcal{I} \approx \mathrm{Fin} \otimes \emptyset$, so it is not a P-ideal.

Summable ideals

Definition

Let $f : \mathbb{N} \to [0, \infty)$ be such that $\sum_{n \in \mathbb{N}} f(n) = \infty$. Then $\mathcal{I}_f = \{B \subseteq \mathbb{N} : \sum_{n \in B} f(n) < \infty\}$ is an ideal called *summable ideal*.

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Theorem (Filipów-Tryba)

If $\lim_{n\to\infty} f(n) = 0$, *then* $\mathcal{I}_f \notin REG$.

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Theorem (Filipów-Tryba)

If $\lim_{n\to\infty} f(n) = 0$*, then* $\mathcal{I}_f \notin REG$*.*

Theorem

$\mathcal{I}_f \in SR2$ *for every summable ideal since* $\mathcal{I}_f = \mathcal{I}(A)$ *for*

$$A = \begin{pmatrix} f(1) & f(2) & f(3) & \cdots \\ 0 & f(2) & f(3) & \cdots \\ 0 & 0 & f(3) & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

Extendability to summable ideals

Theorem

Let $\mathcal{I} = \mathcal{I}(A)$ for some semiregular matrix A. Then $\mathcal{I} \subseteq \mathcal{I}_f$ for some summable ideal $\mathcal{I}_f$.

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Let $\mathcal{I} = \mathcal{I}(A)$ for some semiregular matrix A. Then $\mathcal{I} \subseteq \mathcal{I}_f$ for some summable ideal $\mathcal{I}_f$.

Corollary

$\mathcal{I}_d \notin SR1$ and $\mathcal{I}_d \notin SR2$.

Final results

Theorem

If $\mathcal{I} \in$ REG and $\mathcal{I} \subseteq \mathcal{I}_f$ for some summable ideal $\mathcal{I}_f$, then $\mathcal{I} \in$ SR2.

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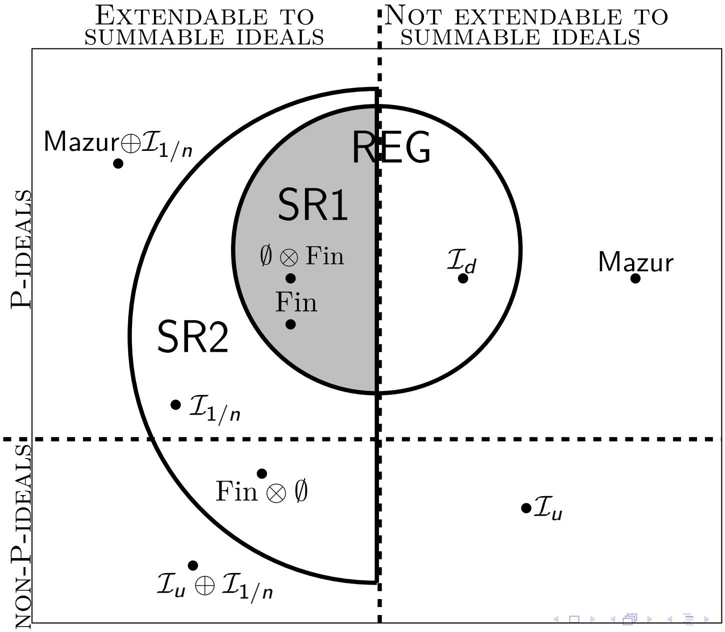
Theorem

$\mathcal{I} \in$ SR1 if and only if $\mathcal{I} \in$ REG and $\mathcal{I} \in$ SR2.

Summary



Extendable to summable ideals
Not extendable to summable ideals
P-ideals
non-P-ideals
Mazur⊕$\mathcal{I}_{1/n}$
REG
SR1
$\emptyset \otimes$ Fin
Fin
SR2
$\mathcal{I}_d$
Mazur
$\mathcal{I}_{1/n}$
Fin $\otimes\ \emptyset$
$\mathcal{I}_u$
$\mathcal{I}_u \oplus \mathcal{I}_{1/n}$