On the existence result for a system of nonlinear equations

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Using the general result about the continuous dependence on parameters for fixed points obtained via the Banach Contraction Principle and next the Schauder fixed point theorem, Avramescu in [1] proved what follows:

Theorem 1 *Let (D_1, d) be a complete metric space, D_2 a closed convex subset of a normed space $(Y,\cdot)$, and let $N_i : D_1 \times D_2 \to D_i$, $i = 1, 2$ be continuous mappings. Assume that the following conditions are satisfied:*
(a) There is a constant $L \in [0, 1)$, such that:

$$d(N_1(x, y), N_1(\overline{x}, y)) \leq Ld(x, x) \text{ for all } x, \overline{x} \in D_1 \text{ and } y \in D_2;$$

(b) $N_2(D_1 \times D_2)$ is a relatively compact subset of Y.
Then, there exists $(x, y) \in D_1 \times D_2$ with:

$$N_1(x, y) = x,\ N_2(x, y) = y.$$

Many authors worked on the above result using various existence principles instead of the Schauder Theorem. Recently, Precup at al. in [3] has imposed partial variational structure on the second mapping and as result instead of the Schauder Theorem applied the Ekeland Variational Principle. Our aim is to look at the above theorem from some other point of view, i.e. to get rid of the usage of the Banach Contraction Principle for the first equation, and to retain some fixed point technique for the second one. In doing so we employ recent parametric dependent version of the Browder-Minty Theorem from [2].

References

[1] Avramescu, C.: On a fixed point theorem (in Romanian). St. Cerc. Mat. 22(2), 215–221 (1970).

[2] M. Bełdziński, M. Galewski, I. Kossowski, Dependence on parameters for nonlinear equations − abstract principles and applications, to appear in Math. Methods Appl. Sci., 10.1002/mma.7882.

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